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An Improvement of the Finite-Element Series-Expansion Technique for Linear Water Waves

F. MATTIOLI

Dipartimento di Fisica, Settore di Geofisica, Universita' di Bologna, Italy

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Recently, the author has introduced a numerical method to deal with the propagation of surface water waves in the framework of the linear theory for an inviscid fluid and simple harmonic motion. The method, based on a finite element scheme, relies on the use of an original series expansion technique of the involved unknown fields to compute the element equations. In this work the effects of some improvements of the method are presented. Some examples show that the method can be applied to the propagation of short waves in very large regions of sea, equal to several hundreds of square wavelengths, where various diffraction, refraction, and reflection phenomena can interact simultaneously in a complicated way.

1. INTRODUCTION

The numerical models dealing with the propagation of surface water waves in cases in which the diffraction and multiple reflection phenomena are predominant are mainly based on the so-called "mild-slope equation" introduced by Berkhoff [1] and Smith and Sprinks [2], also capable of considering refraction phenomena, provided the slope of the bottom is sufficiently small. This equation applies to small amplitude waves in an inviscid fluid, when the motion is assumed to be irrotational and harmonic in time, so that it can be used in a large variety of problems, like the diffraction of waves by a submerged shoal or by a breakwater, or to the wave-induced oscillations in a harbor basin.

The possibilities of the numerical approximations of this equation, however, are limited. In fact, whatever the assumed discretization process, the continuous original equation is reduced to the solution of a system of algebraic equations, whose dimension increases with the size of the region of sea to be modelled. Since a matrix cannot be inverted beyond certain limits, related to the possibilities of present-day computers, it follows that the discretized area of sea cannot exceed a certain number of square wavelengths. For this reason the equation is generally used only for problems defined in regions of at most a few dozen square wavelengths [3, 4], or as a basis to deduce parabolic approximations [5, 6], in which the assumption is made that the wave propagation occurs mainly in a well

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defined direction. In principle the equation might be solved even in larger regions of sea, but at the expense of a remarkable amount of computational effort. We remember that the memory area and the computer time required in the inversion of a matrix grow with the square and the cube of its dimensions, respectively.

Recently, Mattioli [7] has devised a new kind of technique, based on a finite element scheme, in which the original domain is subdivided into elements of arbitrary shape and size, while being of constant depth. Inside each element the unknown field is expanded in terms of particular solutions of the Helmholtz equation, while along the sides of the elements it is expanded in terms of Legendre polynomials. Such a procedure allows one to use few parameters to describe a wavelength, so that it is possible to consider the wave propagation in domains of large extension, of the order of several hundreds of square wavelengths, using only a few thousands of unknowns. The possibilities of the method have suggested continuing the analysis of its properties. The present study shows that most of the numerical problems pointed out in [7] have been solved and that the performance of the method is even better than one could have expected.

2. The Original Equations

The linear theory for an inviscid fluid and irrotational motion reduces the evaluation of the quantities of interest, that is, the surface wave elevation and the velocity field, to the evaluation of the potential field $\Phi(x, y, z, t)$. For monochromatic waves and constant depth, the potential field can be factorized as follows

$$\Phi(x, y, z, t) = \varphi(x, y)\, Z(z)\, e^{-i\omega t},$$

where $\varphi(x, y)$ is the value of potential at the free surface, and $Z(z)$ is the vertical behavior

$$Z(z) = \frac{\cosh k(h+z)}{\cosh kh},$$

where h is the depth and k the wavenumber. On the other hand, the potential φ satisfies the Helmholtz equation

$$\nabla^2\varphi + k^2\varphi = 0 \tag{2.1}$$

with boundary conditions of zero normal derivative along the solid contours of the domain and suitable radiation conditions at infinity.

When the depth in the original problem is variable, it is necessary to subdivide the whole domain into elements, in each of which the depth is assumed to be constant. Then, inside the elements (2.1) will be supposed to be valid, while between the elements the continuity of the elevation field and the flux of the normal velocity will be assumed. That is, if χ represents the flux of the normal derivative of the potential field

$$\chi = b\,\partial_n \varphi,$$

where

$$b = \int_{-h}^{0} Z(z)\,dz,$$

the conditions between two elements will read

$$\varphi_1 = \varphi_2, \tag{2.2}$$

$$\chi_1 = -\chi_2, \tag{2.3}$$

if the index 1 and 2 refer to two adjacent elements.

The approximations (2.1)–(2.3) are certainly valid in the limit of long waves, and should still be a good representation of the reality for intermediate depth, provided the difference in depth between two adjacent elements is small, both with regard to the local depth and to the considered wavelength. Note, for example, that the model neglects the evanescent modes, so that abrupt changes of depth certainly cannot be taken into account. The topic is discussed in Mattioli [7], although a thorough study, stating the limits of these assumptions and a comparison with other possible matching conditions is still lacking.

3. Solution Procedure

Green's functions theory applied to the Helmholtz equation in a domain of limited extension states that, except that in correspondence to the wavenumbers relative to the Neumann eigenfunctions, the normal derivative of the field and the field itself along the contour $\mathscr{C}$ of the domain can be related to each other by an integral equation. A similar relation holds for an unlimited domain, provided the field satisfies the Sommerfeld radiation condition at infinity and the contour $\mathscr{C}$ is the finite part of the boundary in which the normal derivative of the field is different from zero. In both cases, if $\mathscr{G}(\mathbf{x}, \mathbf{x}_0)$ is the Green's function subject to Neumann boundary conditions, it can be written as follows

$$\varphi(\mathbf{x}) = \int_{\mathscr{C}} \mathscr{G}(\mathbf{x}, \mathbf{x}_0)\,\chi(\mathbf{x}_0)/b\,ds_0 + \eta\varphi_{\mathrm{P}}(\mathbf{x}), \tag{3.1}$$

where φ_{P} represents the possible plane wave in an external element, with normal derivative equal to 0 along $\mathscr{C}$, while η is 1 for external elements and 0 for internal elements.

If the analytical expression of the Green's function of all the elements were known, then the solution procedure would be simple. In this case one can choose the flux χ, suitably oriented along the various sides of the finite element network, as main unknown of the problem, and then write along the same sides the equations stating the identity of the elevation field when computed in two adjacent elements. Thus, it is possible to obtain a well-posed system of integral equations, that can be reduced to a system of algebraic equations, once discretized following whatever reasonable procedure. In turn, the solutions of this system can be inserted in the discrete versions of (3.1) to achieve an approximate version of the elevation field along the sides of the finite element mesh.

The method of solution devised by Mattioli [7] provides an efficient tool to compute an approximate version of the Green's function for elements of arbitrary shape and size. It relies on a series expansion of the internal field in terms of particular solutions of the Helmholtz equation, which shows to be quickly convergent to the exact field. Furthermore, the surface potential field and its normal derivative are expanded along the contour of the elements in terms of Legendre polynomials, in such a way to keep the number of parameters needed to describe a wavelength as low as possible.

Thus the solution of any problem is reduced to the evaluation of a discrete version of the continuous equations written for each element, after which the various element matrices and known-term vectors must be assembled to obtain the global system of equations in the same way followed in the traditional finite element method. More details of the procedure can be found in the quoted work [7].

4. The Method of Numerical Integration

By adopting the repeated mid-point rule to evaluate the various integrals present in the previously described formulation, the results show discontinuous, even divergent trends sometimes, near the ends of the sides of the finite element mesh. Moreover, by increasing the number of the integration intervals, these divergent trends do not disappear, but only reduce to narrower intervals. So, Mattioli [7] had to resort to hand-interpolation to obtain reliable results, while the suspicion of a nonuniform convergence of the method remained.

It has been possible, however, to establish that such a behavior of the solution depends to large extent on the integration method used in the internal elements. In fact, by adopting the Gauss–Legendre method, with a sufficiently high number of sample points, the solutions become continuous and much more accurate than before. Furthermore, there is a not negligible saving of computational time, because the number of the times in which the Bessel functions or the Legendre polynomials must be evaluated in strongly reduced.

However, once this drastic improvement has been obtained, some small discontinuities still remain at the ends of some sides belonging to the lines joining the internal and half-plane elements, where the previous integration method was employed. For these elements, in fact, the Green's function

$$\mathscr{H}(s, s_0) = -\tfrac{1}{2}\imath H_0^{(1)}(k\,|s-s_0|)$$

shows a logarithmic singularity for $s \to s_0$, so that Gauss–Legendre formulae cannot directly be applied. It is enough, however, to subtract the singularity, and then integrate it analytically. In essence, it is necessary to evaluate the following double integral [7]

$$\int_{-1}^{1} P_n(s)\,ds \int_{-1}^{1} \lg \tfrac{1}{2}\,|s-s_0|\; P_m(s_0)\,ds_0 = \begin{cases} 0 & \text{if } n+m \text{ odd,} \\ c_{nm} & \text{if } n+m \text{ even,} \end{cases} \tag{4.1}$$

where $P_n(s)$ is the Legendre polynomial of order n.

It has been found that c_{nm} is a symmetric matrix of inverses of integer numbers, except $c_{00} = -6$. In the Appendix some details are given about the computations which have been performed, along with the matrix of the numbers $d_{nm} = 1/c_{nm}$.

This modification leads to a further improvement of the results, which are now never discontinuous at any point of the lines of the finite element network.

5. Critical Wavenumbers

It is known [7, 8] that two series of critical wavenumbers exist, corresponding to the Neumann and Dirichlet eigenfunctions of the element, for which the element equations no longer have a unique solution. For example, in a circular element of radius R the matrix $\mathbf{W}$ [7, Sect. 4; 8, Sect. 15.3] takes the form

$$W_{\left\{\substack{2n+1,2m+1 \\ 2n+2,2m+2}\right\}} = \int_0^{2\pi} J_n(kR) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} kRJ'_m(kR) \begin{Bmatrix} \cos m\theta \\ \sin m\theta \end{Bmatrix} d\theta$$

$$= \tfrac{1}{2}\pi \begin{Bmatrix} \varepsilon_n \\ 1 \end{Bmatrix} J_n(kR)\, kRJ'_n(kR)\, \delta_{nm}, \qquad n, m = 0, 1,\ldots,$$

$$W_{\left\{\substack{2n+1,2m+2 \\ 2n+2,2m+1}\right\}} = 0, \qquad n, m = 0, 1,\ldots,$$

if the solutions of the Helmholtz equation are ordered as follows

$$J_0(kr),\ J_1(kr)\cos\theta,\ J_1(kr)\sin\theta,\ldots,$$

where J_n, $n = 0, 1,\ldots$, are the Bessel function of the first kind and order n, and $\varepsilon_0 = 2$, $\varepsilon_n = 1$, $n > 0$.

Hence, the diagonal matrix $\mathbf{W}$ becomes singular in correspondence with the zeros of $J_n(kR)$ and $J'_n(kR)$ that is, in correspondence with the two classes of eigenfunctions previously quoted. The possibility of critical wavenumbers is completely excluded only if the element is sufficiently small in comparison to the wavelength (in this case $kR < 2.4$). However, the convenience of an element increases with its size, so this property is not of great help.

The results obtained in various general-purpose tests have shown that the error is a smooth function of the wavenumber, so that the typical divergent behaviors of the error in the neighborhood of the critical wavenumbers were not coming up. A more detailed analysis has been carried out for a square element with $K = 6$ parameters per side around

$$ka = \{(2\pi)^2 + \pi^2\}^{1/2} = 7.0248,$$

that is, for the same critical wavenumber treated in [7]. In correspondence to this wavenumber we have two Neumann eigenfunctions with two nodal lines parallel to a side and one nodal line orthogonal to them, and two Dirichlet eigenfunctions with one internal nodal line parallel to a side. When the repeated mid-point rule is adopted as integration method, the error curve shows a divergent behavior around the critical wavenumbers, limited to the range $7.0247 < ka < 7.0249$, that is, to an interval equal to about $3 \cdot 10^{-5}$ times the wavenumber. With the more approximate Gauss–Legendre integration the deviation of the error of its ordinary value is negligible when a step equal to 10^{-4} is adopted. Hence, if critical wavenumber intervals exist, they are very narrow, and in any case they might be avoided by simply changing the considered wavelength by a small amount.

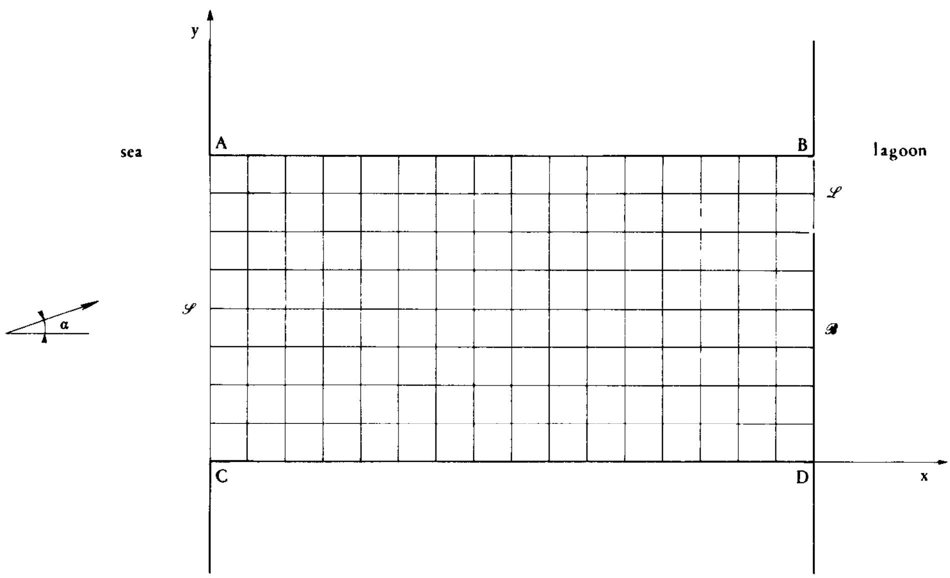

FIG. 1. Sketch of the channel connecting the sea to the lagoon, partially closed by a barrier. α is the angle of incidence of the plane wave coming from the sea. Both the sea and the lagoon are represented by half-plane domains subject to Sommerfeld radiation conditions.

6. Wave Propagation in a Domain of Large Extension

An application of the method has been carried out for a problem of wave propagation in a layer of water of intermediate depth in a region of sea of large extension. This example is the same as the one considered in Mattioli [7], that is, a channel connecting the sea to a lagoon partially obstructed at the rear by a barrier (Fig. 1), in order to make an evaluation of the obtained improvements possible. The waves coming from the sea, after being refracted by the variable bottom and reflected by the lateral walls of the channel and by the rear barrier, are partially diffracted towards the lagoon, and partially radiated towards the sea. Thus, the resulting field is a combination of different processes, that, in general, can be quantitatively evaluated only by a numerical model.

The channel width has been chosen as one-half of the length L, and the barrier closes the access to the lagoon for three quarters of the length. All the contours have been assumed to be perfectly reflective and the depth has been supposed to be constant, to allow a comparison with the practically exact results of Mattioli and Tinti [9].

To consider a wave for which $kL = 250$, that is of length equal to $\frac{1}{20}$ of the channel width, the domain has been subdivided into 8×16 square elements, and 11 unknown parameters have been attributed to each side. This means that each square element contains about $2.49 \times 2.49 = 6.18$ square wavelengths. Note that for $L = 2$ Km and $h_0 = 10$ m the wavelength is 50 m and the period about 6.16 s. The behavior of the modulus $|A|$ of the amplification factor with regard to the

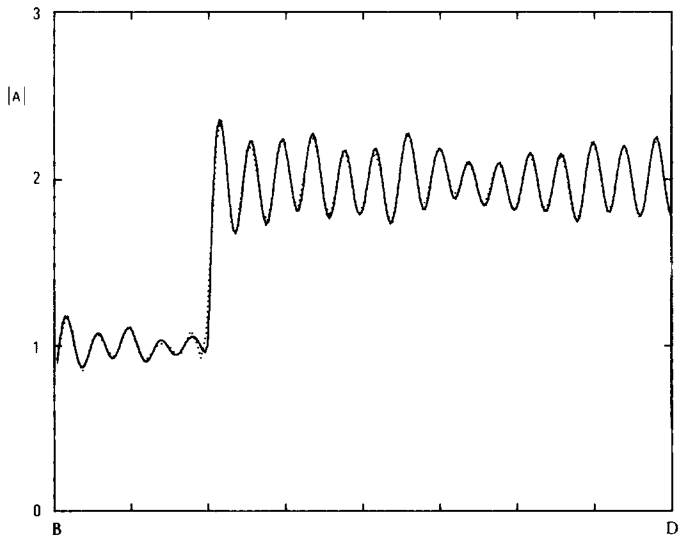

Fig. 2. Modulus of the amplification factor $|A|$ along the line B–D for a channel of constant depth, normal incidence and $kL = 250$. The continuous and dotted lines refer to the numerical and theoretical results, respectively.

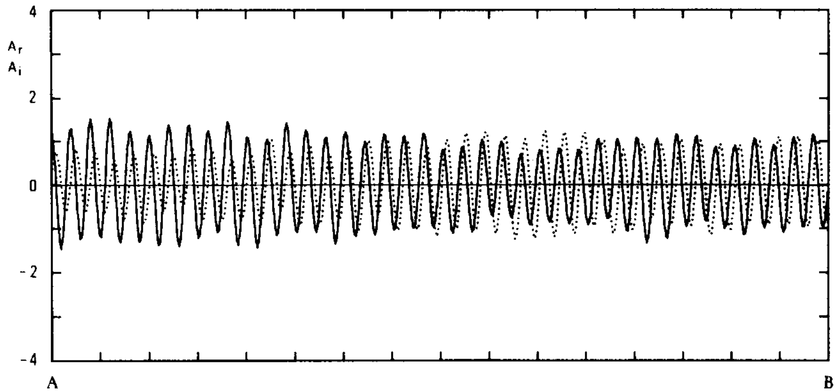

FIG. 3. Real (continuous line) and imaginary (dotted line) parts A_r and A_i of the amplification factor along the channel wall A–B in the same conditions of Fig. 2.

amplitude of the incoming wave along line B–D for a constant depth and normal incidence is shown in Fig. 2, and is compared with the exact solution. As can be seen the agreement is excellent, although fewer parameters (only 3080) have been used in comparison with Mattioli [7]. This means that only 3.85 parameters per square wavelength have been proved to be sufficient to obtain a very high accuracy in the results. We can recall that an extension of sea equal to $20 \times 40 = 800$ square wavelengths can be faced by the traditional finite element techniques only by the use of some dozen of thousands of nodal variables.

To be aware of the kind of information that can be drawn by a program like this, the real and the imaginary parts A_r and A_i of the amplification field along the walls A–B and C–D of the channel have been plotted in Fig. 3 and Fig. 4. In the former

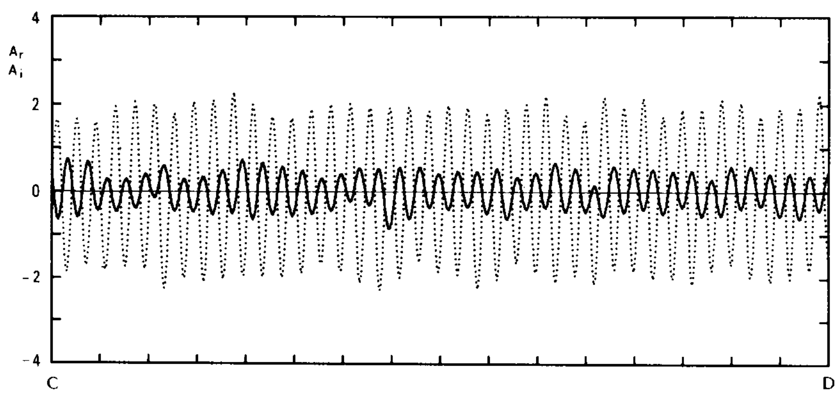

FIG. 4. As in Fig. 3, but with reference to the channel wall C–D.

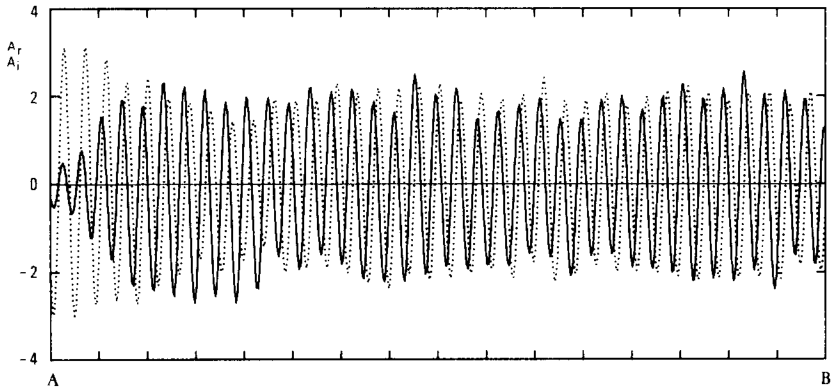

FIG. 5. Real and imaginary parts of the amplification factor along A–B for a channel of variable depth, angle of incidence $\alpha = 22.5°$ and $kL = 250$.

case it is possible to detect the typical behavior of a progressive wave from the sea to the lagoon, with the two components out of phase by 90°. In the latter the two components are in opposition of phase, indicating a standing wave between the barrier and the sea. The information is very detailed, and can be of great help when the dynamics of the wave motion is no longer so simple as in this case.

In Fig. 5 and Fig. 6 the amplification fields along the same contours A–B and C–D are shown when the waves incide with an angle of 22.5° with regard to the axis of the channel and the depth is variable as in Mattioli [7, Eq. (5.1)]. Observe the different phase delays in the figures, indicating a wave mainly propagating from the sea to the lagoon in the former case, and in the opposite direction in the latter, as is foreseeable by elementary considerations. Instead, the intensity of the amplitude modulations, especially in the lower wall of the channel, are clearly not evaluable by other means than with a numerical model.

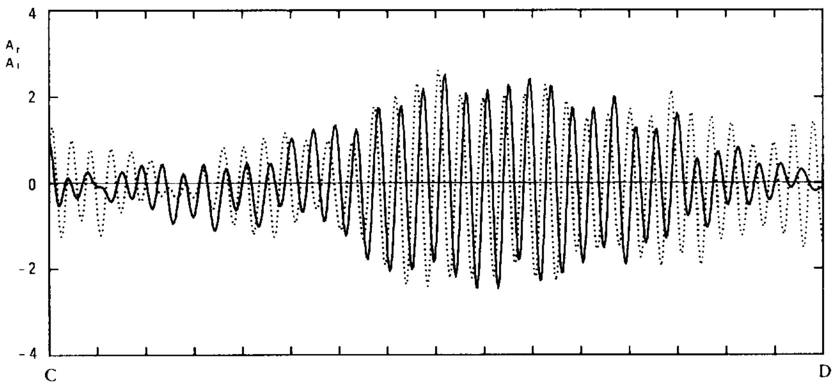

FIG. 6. The same as before, but along the wall C–D.

7. Conclusion

The most striking result of the method presented here consists in the possibility of using a very small number of parameters to describe one wavelength. As a consequence, it is possible to consider phenomena of wave propagation in larger regions of sea, by an order of magnitude or more, than the ones considered by the known methods, when the same total number of variables is used, and further improvements are not to be excluded. In fact, with a CDC 7600, neither has the maximum memory area been occupied, nor has the computer time, which is of the order of 100 s in the largest applications essentially equally divided between the construction of the element matrices and the inversion of the global matrix, are prohibitive. Moreover, the program has not been completely optimized, although several contrivances have been employed, at least with regard to the previous version.

The adoption of a more accurate integration rule makes the solution smooth in every point of the domain, so that the problem of discontinuous outputs pointed out in [7] is eliminated. Furthermore the method does not seem to be affected by the presence of critical wavenumbers, that do not give rise to detectable variations of the numerical error, at least when an adequate distribution of the parameters is adopted. In conclusion, we can say that the most important numerical problems set in Mattioli [7] have been satisfactorily solved.

To sum up, the research performed so far shows that a very powerful numerical method for solving the Helmholtz equation has been devised, and that several developments, which will be considered in the near future, should be possible along this line of research.

Appendix

The kernel of the integral operator representing a half-plane element becomes singular in two circumstances: when s and s_0 belong to the same side and $s \to s_0$, and when s and s_0 belong to adjacent sides, and both tend to the common vertex.

Let us consider the first case. When $n+m$ is odd, then (4.1) simplyfies to

$$2\int_{-1}^{1} P_n(s)\,ds \int_{-1}^{s} \lg \tfrac{1}{2}(s-s_0)\, P_m(s_0)\, ds_0 = c_{nm}.$$

To calculate this integral the program for algebraic manipulation SCHOONSCHIP has been used. The computations have been carried out according the following steps:

— Substitution $(s-s_0) \to v$.

— Substitution

$$\int_0^{1+s} v^n \lg\frac{1}{2}\, v\, dv = \frac{(1+s)^{n+1}}{n+1}\left[\lg\frac{1}{2}(1+s) - \frac{1}{n+1}\right].$$

— Evaluation of

$$I_n = \int_{-1}^{1} s^n \lg \tfrac{1}{2}(1+s)\, ds$$

by means of the recursion relation

$$(n+1)\, I_n = K_n - nI_{n-1},$$

where

$$K_n = -\frac{1+(-1)^n}{n+1} \qquad \text{and} \qquad I_0 = K_0 = -2.$$

This identity can be obtained by integrating by parts, with $\lg \frac{1}{2}(1+s)$ as integrand factor. This step is fundamental if one wants to compute the coefficients c_{nm} up to $n, m = 14$. The coefficients $d_{nm} = 1/c_{nm}$ for $n+m>0$ are given in Table I.

In the second case, referring to two adjacent sides, we can recall that $P_n(-1) = (-1)^n$, so that in order to eliminate the singularity at the point $(s, s_0) = (1, 1)$, it merely suffices to compute

$$\int_{-1}^{1} ds \int_{1}^{3} \lg \tfrac{1}{2}(s-s_0)(-1)^m\, ds_0 = (-1)^m\,(-8+8\lg 2).$$

TABLE I

Table of the Coefficients $d_{nm} = 1/c_{nm}$, $n = 0, 1, \ldots, 14$, $m = 1, \ldots, 14$

n \ m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0		3		45		210		630		1485		3003		5460
1	−1		9		90		350		945		2079		4004	
2		−3		18		150		525		1323		2772		5148
3			−6		30		225		735		1764		3564	
4				−10		45		315		980		2268		4455
5					−15		63		420		1260		2853	
6						−21		84		540		1575		3465
7							−28		108		675		1925	
8								−36		135		825		2310
9									−45		165		990	
10										−55		198		1170
11											−66		234	
12												−78		273
13													−91	
14														−105

Note. Recall that the matrix of the coefficients is symmetric, that for $n+m$ odd $c_{nm} = 0$, and that $c_{00} = -6$.

This correction, in spite of its simplicity, shows itself to be very important in eliminating the last sources of error.

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REFERENCES

1. J. C. W. BERKHOFF, in *Proc. 13th Coastal Engng Conf., Vancouver*, ASCE, Vol. 1 (1972), p. 471.
2. R. SMITH AND T. SPRINKS, *J. Fluid Mech.* **72**, 373 (1975).
3. H. S. CHEN AND C. C. MEI, Ralph M. Parsos Laboratory for Water Resources and Hydrodynamics, Report No. 190 (1974).
4. J. R. HOUSTON, *Appl. Ocean Res.* **3**, 163 (1981).
5. A. C. RADDER, *J. Fluid Mech.* **95**, 159 (1979).
6. T.-K. TSAY AND P. L.-F. LIU, *J. Geophys. Res.* **87**, 7932 (1982).
7. F. MATTIOLI, *Comput. Fluids* **13**, 119 (1985).
8. F. MATTIOLI, *Nuovo Cimento C* **7**, 81 (1984).
9. F. MATTIOLI AND S. TINTI, *Nuovo Cimento C* **1**, 31 (1978).